Without using L'Hôpital rule, find the following limits.

(a) $\lim_{x \to 1} \frac{x^2 - 3x + 2}{x^2 - 1}$ $\quad \frac{(x-2)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}}$

(b) $\lim_{x \to 2} \frac{x^3 - 4x^2 + 5x - 2}{x^2 - 4}$

(c) $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$

(d) $\lim_{x \to 27} \frac{\sqrt[3]{x} - 3}{x - 27}$

(e) $\lim_{x \to 3} \frac{x - 3}{\sqrt{x+1} - 2}$

(f) $\lim_{x \to 0} \frac{\sqrt{1 + x^2} - 1}{x}$

(a) $\lim_{x \to -\infty} e^{1+x^6}$;

(b) $\lim_{x \to +\infty} \ln\left(e^{-2x} + e^{-x} + 1\right)$;

(c) $\lim_{x \to +\infty} \ln\left(\frac{e^{3x} + e^{x}}{e^{5x} + e^{2x}}\right)$;

(d) $\lim_{x \to +\infty} \ln\left(\frac{e^{2x+1} + 2e^{-x}}{e^{2x} + e^{-x+2}}\right)$;

13. By using the fact that $\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$, find the following limits

(a) $\lim_{x \to +\infty} \left(1 + \frac{2}{x}\right)^{2x}$

(b) $\lim_{x \to +\infty} \left(1 + \frac{1}{x+1}\right)^{x}$

(c) $\lim_{x \to +\infty} \left(\frac{x}{x-1}\right)^{x}$

5. Let $f(x) = \frac{|x-1|}{x^2 - 1}$ for $x \neq$

(a) Does $\lim_{x \to 1} f(x)$ exist?

(b) Does $\lim_{x \to -1} f(x)$ exist?

9. By using the fact that $\lim_{x \to 0} \frac{\sin x}{x} = 1$, find the following limits.

(a) $\lim_{x \to 0} \frac{\sin 2x}{5x}$

(b) $\lim_{x \to 0} \frac{\sin 5x}{\sin 7x}$

(c) $\lim_{x \to 0} \frac{\sin(x^2)}{5x^2}$

(d) $\lim_{x \to 0} \frac{\cos ax - \cos bx}{x^2}$, where a and b are distinct real numbers.

14. Without using L'Hôpital rule, find the following limits, if exist.

(a) $\lim_{x \to +\infty} \frac{x^2 - 3x + 2}{x^2 - 1}$

(b) $\lim_{x \to -\infty} \frac{x^3 - 2x}{4x^3 + 2x^2}$

(c) $\lim_{x \to +\infty} \frac{\sqrt{x^2 + 4}}{x + 4}$

(d) $\lim_{x \to +\infty} \frac{x}{\sqrt{9x^2 + 5}}$

(e) $\lim_{x \to -\infty} \frac{x}{\sqrt{9x^2 + 5}}$

(f) $\lim_{x \to +\infty} \sqrt{x+1} - \sqrt{x-1}$

(g) $\lim_{x \to +\infty} \sqrt{x^2 + x} - x$

(h) $\lim_{x \to -\infty} \sqrt{x^2 + x} - x$

Without using L'Hôpital rule, find the following limits.

(a) $\lim\limits_{x\to1} \dfrac{x^2-3x+2}{x^2-1}$

(b) $\lim\limits_{x\to2} \dfrac{x^3-4x^2+5x-2}{x^2-4}$

(c) $\lim\limits_{x\to4} \dfrac{\sqrt{x}-2}{x-4}$

(d) $\lim\limits_{x\to27} \dfrac{\sqrt[3]{x}-3}{x-27} = \lim\limits_{x\to27} \dfrac{\cancel{\sqrt[3]{x}-3}}{\cancel{(\sqrt[3]{x}-3)}(x^{\frac{2}{3}}+3x^{\frac{1}{3}}+9)}$

(e) $\lim\limits_{x\to3} \dfrac{x-3}{\sqrt{x+1}-2}$

(f) $\lim\limits_{x\to0} \dfrac{\sqrt{1+x^2}-1}{x}$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

$= \lim\limits_{x\to3} \dfrac{x-3}{\sqrt{x+1}-2}\cdot\dfrac{\sqrt{x+1}+2}{\sqrt{x+1}+2}$

$\dfrac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}+1}$

$= \lim\limits_{x\to3} \dfrac{(x-3)(\sqrt{x+1}+2)}{x+1-4} = \lim\limits_{x\to3} \sqrt{x+1}+2 = 4$

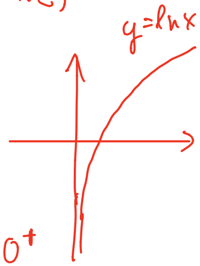
(a) $\lim_{x\to-\infty} e^{1+x^6}$; $= \infty$ (DNE)

$\lim_{y\to\infty} e^y = \infty$ (DNE)

(b) $\lim_{x\to+\infty} \ln\left(e^{-2x}+e^{-x}+1\right)$; $= \ln(0+0+1)$

$\lim_{y\to-\infty} e^y = 0$

(c) $\lim_{x\to+\infty} \ln\left(\dfrac{e^{3x}+e^{x}}{e^{5x}+e^{2x}}\right)$; $= \lim_{x\to\infty} \ln\left(\dfrac{e^{-2x}+e^{-4x}}{1+e^{-3x}}\right) = -\infty$



(d) $\lim_{x\to+\infty} \ln\left(\dfrac{e^{2x+1}+2e^{-x}}{e^{2x}+e^{-x+2}}\right)$;

As $x\to\infty$, $\dfrac{e^{-2x}+e^{-4x}}{1+e^{-3x}} \to 0^+$

$\lim_{x\to+\infty} \ln \dfrac{e+2e^{-3x}}{1+e^{-3x+2}} = \ln\dfrac{e+0}{1+0} = 1$

9. By using the fact that $\lim_{x\to 0}\dfrac{\sin x}{x} = 1$, find the following limits.

(a) $\lim_{x\to 0}\dfrac{\sin 2x}{5x}$ $= \lim_{x\to 0} \dfrac{\sin 2x}{2x}\cdot\dfrac{2}{5} = (1)\dfrac{2}{5} = \dfrac{2}{5}$

(b) $\lim_{x\to 0}\dfrac{\sin 5x}{\sin 7x}$ $= \lim_{x\to 0} \dfrac{\sin 5x}{5x}\cdot\dfrac{7x}{\sin 7x}\cdot\dfrac{5}{7} = \dfrac{5}{7}$

(c) $\lim_{x\to 0}\dfrac{\sin(x^2)}{5x^2}$ $= \dfrac{1}{5}$

(d) $\lim_{x\to 0}\dfrac{\cos ax-\cos bx}{x^2}$, where a and b are distinct real numbers.

$= \lim_{x\to 0}\dfrac{-a\sin ax + b\sin bx}{2x} = \lim_{x\to 0}\dfrac{-a^2\cos ax + b^2\cos bx}{2}$

$= \dfrac{b^2-a^2}{2}$

13. By using the fact that $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$, find the following limits.

(a) $\lim_{x\to+\infty}\left(1+\frac{2}{x}\right)^{2x}$ $= \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{2}}\right)^{\frac{x}{2}\cdot 4} = \left[\lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{2}}\right)^{\frac{x}{2}}\right]^4 = e^4$

(b) $\lim_{x\to+\infty}\left(1+\frac{1}{x+1}\right)^x$

(c) $\lim_{x\to+\infty}\left(\frac{x}{x-1}\right)^x$ $\leftarrow$ let $y=\left(\frac{x}{x-1}\right)^x$ $\ln y = x\ln\frac{x}{x-1}$

$1+\frac{1}{x-1}$

$\lim_{x\to\infty}\ln y = \lim_{x\to\infty} x\ln\frac{x}{x-1}$

5. Let $f(x) = \frac{|x-1|}{x^2-1}$ for $x \neq \pm 1$.

(a) Does $\lim_{x\to 1} f(x)$ exist?

(b) Does $\lim_{x\to -1} f(x)$ exist?

$|y| = \begin{cases} y & \text{if } y \geq 0 \\ -y & \text{if } y<0 \end{cases}$

5a. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}\frac{|x-1|}{x^2-1} = \lim_{x\to 1^+}\frac{x-1}{x^2-1} = \lim_{x\to 1^+}\frac{1}{x+1} = \frac{1}{2}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\frac{|x-1|}{x^2-1} = \lim_{x\to 1^-}\frac{-(x-1)}{x^2-1} = \lim_{x\to 1^-}\frac{-1}{x+1} = -\frac{1}{2} \neq \lim_{x\to 1^+} f(x)$

$\therefore \lim_{x\to 1} f(x)$ DNE

b. $\lim_{x\to -1} f(x)$ DNE

14. Without using L'Hôpital rule, find the following limits, if exist.

(a) $\displaystyle\lim_{x\to+\infty} \frac{x^2-3x+2}{x^2-1} = \lim_{x\to+\infty} \frac{1-\frac{3}{x}+\frac{2}{x^2}}{1-\frac{1}{x^2}} = \frac{1}{1} = 1$

(b) $\displaystyle\lim_{x\to-\infty} \frac{x^3-2x}{4x^3+2x^2} = \frac{1}{4}$

(c) $\displaystyle\lim_{x\to+\infty} \frac{\sqrt{x^2+4}}{x+4} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{4}{x^2}}}{1+\frac{4}{x}} = \frac{1}{1}$

(d) $\displaystyle\lim_{x\to+\infty} \frac{x}{\sqrt{9x^2+5}}$

(e) $\displaystyle\lim_{x\to-\infty} \frac{x}{\sqrt{9x^2+5}}$

$$\lim_{x\to-\infty} \frac{x}{\sqrt{9x^2+5}} = \lim_{x\to-\infty} \frac{\frac{x}{\sqrt{x^2}}}{\frac{\sqrt{9x^2+5}}{\sqrt{x^2}}} = \lim_{x\to-\infty} \frac{\frac{x}{|x|}}{\sqrt{\frac{9x^2+5}{x^2}}} = \lim_{x\to-\infty} \frac{\frac{x}{-x}}{\sqrt{9+\frac{5}{x^2}}} = -\frac{1}{3}$$

(f) $\displaystyle\lim_{x\to+\infty} \sqrt{x+1}-\sqrt{x-1}$

(g) $\displaystyle\lim_{x\to+\infty} \sqrt{x^2+x}-x$ Rationalization

(h) $\displaystyle\lim_{x\to-\infty} \underbrace{\sqrt{x^2+x}}_{\to\infty}\underbrace{-x}_{\to\infty} = +\infty$ (DONE)

6. Let a be a real number and let $f(x)$ be a function defined by

$$f(x) = \begin{cases} e^x & \text{if} \quad x > 0, \\ 1 & \text{if} \quad x = 0, \\ \cos x & \text{if} \quad x < 0. \end{cases}$$

(a) Find $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$.

(b) Is $f(x)$ is continuous at $x = 0$?

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \cos x = 1$ ← equal

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^x = e^0 = 1$ ←

$\lim_{x\to 0} f(x) = 1$

b. $\lim_{x\to 0} f(x) = 1 = f(0)$

$\Rightarrow f$ is continuous at 0

12. Show that the equation $4^x = 3^x + 2^x$ has at least one solution.

IVT

MVT

$4^x = 3^x + 2^x$

let $f(x) = 4^x - 3^x - 2^x$

$f(0) = -1 < 0$

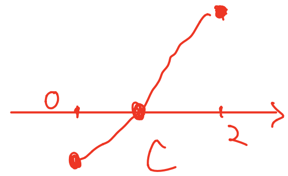
$f(2) = 16 - 9 - 4 = 3 > 0$

Also, f is continuous on $[0,2]$

By IVT, $\exists c \in (0,2)$ s.t.

$f(c) = 0$

$4^c = 3^c + 2^c$



$\therefore x = c$ is a solution

$(e^x)' = e^x$

Find the first derivatives of the following functions.

(a) $y = 2x^3 - 4x + 2$

(b) $y = 5x^3 - 4x^2 + 7$

(c) $y = e^{3x}$ $\quad y' = e^{3x}(3x)' = 3e^{3x}$

(d) $y = \cos 2x$ $\quad y' = (-\sin 2x)(2)$

(e) $y = \sin 5x$

(f) $y = -\tan 3x$

(g) $y = \sqrt{x}$

(h) $y = \ln(1 + x^2)$ $\quad y' = \frac{1}{1+x^2}(2x)$

(a) $y = (2x + 1)^3(x - 1)^4\sqrt{(3x + 2)^5}$

(b) $y = \dfrac{e^{2x}}{(x - 1)^4}$ $\quad y' = \dfrac{(x-1)^4(2e^{2x}) - 4(x-1)^3 e^{2x}}{(x-1)^8}$

(c) $y = x^x$ $\quad$ log diff

(d) $y = (\sin x)^{(\cos x)}$

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

7. Find the first derivatives of the following functions.

(a) $y = 4\sqrt{x} + \dfrac{2}{\sqrt{x}}$

(b) $y = x^3e^{-2x}$

(c) $y = \sin x \ln x$

(d) $y = \sec x - 3\tan x$

(e) $y = x \csc x$

(f) $y = \dfrac{3x - 4}{x + 2}$

(g) $y = \dfrac{x^2 + 1}{x + 1}$

(h) $y = \dfrac{\sin x}{x}$

(i) $y = (3x^2 - 4)^{10}$

(j) $y = \sqrt{x^3 + 1}$

(k) $y = \ln(\ln x)$

(l) $y = e^{\cot x}$

(m) $y = \ln(x + \sqrt{x})$

(c) $y = x^x = e^{\ln x^x} = \overset{\text{constant}}{e^{x\ln x}}$

$y' = e^{x\ln x}(x\ln x)' = e^{x\ln x}\left[\ln x + \frac{x}{x}\right) = e^{x\ln x}(1+\ln x)$

$= x^x(1+\ln x)$

(d) $y = (\sin x)^{\cos x}$

$\frac{d}{dx}$ $\quad \ln y = \ln[(\sin x)^{\cos x}] = \cos x \ln(\sin x)$

$\frac{1}{y}y' = -\sin x \ln(\sin x) + \cos x \cdot \frac{\cos x}{\sin x}$

$y' = y\left(-\sin x \ln(\sin x) + \frac{\cos^2 x}{\sin x}\right)$

$\frac{d}{dx}\ln(x^2+x)$

$= \frac{1}{x^2+x}(2x+1)$

Suppose

$$f(x) = \begin{cases} 3 - \sin x & \text{if} \quad x < 0, \\ a & \text{if} \quad x = 0, \\ bx + c & \text{if} \quad x > 0, \end{cases}$$

where a, b are some real numbers. Given that $f(x)$ is continuous at $x = 0$.

(a) What is the values of a and c?

(b) Find $Lf'(0)$.

(c) Find $Rf'(0)$ (in terms of b).

(d) For what value(s) of b is the function $f(x)$ differentiable at 0?

d. f diffentiable at 0 ($f'(0)$ exist)

$$\Rightarrow Lf'(0) = Rf'(0)$$

$$-1 = b$$

(a)

$$f(0) = a$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} bx + c = c$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 3 - \sin x = 3 - \sin 0 = 3$$

f is continuous at 0 $\Rightarrow$ $f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$

$$a = c = 3$$

b. Correct: $Lf'(0) = \lim_{h\to 0^-} \dfrac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \dfrac{3 - \sin h - 3}{h} = \lim_{h\to 0^-} \dfrac{-\sin h}{h} = -1$

$$Rf'(0) = \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{bh + 3 - 3}{h} = b$$

1. Compute the derivative of each of the following function at the given point by using the formal definition (first principle).

   (a) $f(x) = x^2 + 1$ at the point $x = 2$;

   (b) $f(x) = \frac{1}{x}$ at the point $x = 3$; $f'(x) = -\frac{1}{x^2}$ $f'(3) = -\frac{1}{9}$

   (c) $f(x) = \cos x$ at the point $x = \frac{\pi}{2}$;

   (d) ~~(Harder Problem) $f(x) = x^n$, where n is a natural number, at the point $x = 2$.~~

Sol (b)

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3-3-h}{3(3+h)}}{h}$$

$$= \lim_{h\to 0} \frac{-h}{3(3+h)h} = \lim_{h\to 0} \frac{-1}{3(3+h)} = \frac{-1}{3(3+0)} = -\frac{1}{9}$$

15. Approximate the value of $e^{0.1}$ by linearizing an appropriately chosen function at an appropriately chosen point.

$f(0.1)$

$f(x)=e^x \quad f'(x)=e^x$

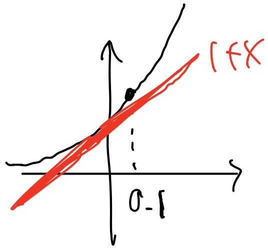
$f(0)=f'(0)=e^0=1$

Sol Let $f(x)=e^x$

Find linearization of $f(x)$ at $a=0$

$$L(x) = f(0)+f'(0)(x-0)$$
$$= 1+1(x) = 1+x$$

$$e^{0.1} = f(0.1)$$
$$\approx L(0.1)$$
$$= 1.1$$



11. Find $\dfrac{dy}{dx}$ in terms of x and y for the following implicit functions.

(a) $x^2 + y^2 = 9$

(b) $x^3y + xy^2 = 1$

(c) $x^3 + y^3 = 2xy$

(d) $ye^{xy} = 1$

$ye^{xy}=1$

$$\frac{dy}{dx}e^{xy} + y\frac{d}{dx}e^{xy} = 0$$

$\dfrac{d^2y}{dx^2}$

12. Let $\mathcal{C}$ be the curve given by the equation $x^3 + xy + y^3 = 11$.

(a) Show that $A = (1, 2)$ is a point lies on $\mathcal{C}$.

(b) Find the equation of tangent of $\mathcal{C}$ at the point A.

$$\frac{dy}{dx}e^{xy} + ye^{xy}\left(y + x\frac{dy}{dx}\right) = 0$$

13. If $y = x^2e^x$, show that $\dfrac{d^2y}{dx^2} = 2\dfrac{dy}{dx} - y + 2e^x$.

$$\frac{dy}{dx}e^{xy} + y^2e^{xy} + xye^{xy}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{y^2}{1+xy}$$

12. Let $\mathcal{C}$ be the curve given by the equation $x^3 + xy + y^3 = 11$.

(a) Show that $A = (1, 2)$ is a point lies on $\mathcal{C}$. ✓

(b) Find the equation of tangent of $\mathcal{C}$ at the point A.

(b) $x^3 + xy + y^3 = 11$

$$3x^2 + y + x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

Put $(x, y) = (1, 2)$

$$3 + 2 + (1)\left.\frac{dy}{dx}\right|_{(1,2)} + 3(4)\left.\frac{dy}{dx}\right|_{(1,2)} = 0$$

$$\left.\frac{dy}{dx}\right|_{(1,2)} = -\frac{5}{13}$$

Eqn of tangent

$$\frac{y-2}{x-1} = -\frac{5}{13}$$

$$y - 2 = -\frac{5}{13}(x-1)$$

$$y = 2 - \frac{5}{13}(x-1)$$

2. Let $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 3$.

(a) Find $f'(x)$. By using the factor theorem or otherwise, show that $f'(x) = 4(x-1)(x-2)(x-3)$.

(b) In the following table, fill in the signs of the factors in the corresponding intervals.

	$x < 1$	$x = 1$	$1 < x < 2$	$x = 2$	$2 < x < 3$	$x = 3$	$x > 3$
$x-1$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-2$							
$x-3$							
$f'(x)$							

(c) Solve $f'(x) > 0$ and $f'(x) < 0$.
Hence, find the extreme points of the graph $y = f(x)$.

6. Let $f(x) = xe^{-x^2}$.

(a) Find $f'(x)$ and $f''(x)$.

(b) Determine the values of x for each of the following cases:

(i) $f'(x) = 0$;	(iii) $f'(x) < 0$;	(v) $f''(x) > 0$;
(ii) $f'(x) > 0$;	(iv) $f''(x) = 0$;	(vi) $f''(x) < 0$.

(c) Find all relative extrema and points of inflexion of $f(x)$.

(d) Sketch the graph of $f(x)$.

3. Let $f(x) = x^2 \ln x$ for $x > 0$.

Find $f'(x)$ and $f''(x)$. Hence, determine the extreme point(s) of the function.

5. Let $f(x) = \dfrac{x^2 + 3x}{x - 1}$.

(a) Find $f'(x)$.

(b) Determine the values of x for each of the following cases:

(i) $f'(x) = 0$; (ii) $f'(x) > 0$; (iii) $f'(x) < 0$.

(c) Find all relative extrema of $f(x)$.

(d) Find all asymptotes of $f(x)$.

(e) Sketch the graph of $f(x)$.

4. Find the greatest and least values of the following functions on the given closed interval:

(a) $f(x) = x - 2\sqrt{x}$ on $[0, 9]$;

(b) $f(x) = x^4 - 8x^2 + 2$ on $[-1, 3]$;

(c) $f(x) = e^x \ln x$ on $[1, 2]$.

9. By using the mean value theorem, prove that for all $x, y \in \mathbb{R}$,

$$\cancel{|\cos x - \cos y| \le |x - y|}.$$

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$a < c < b$$

10. By using the mean value theorem, prove that for all $x > 0$,

$$1 + x < e^x < 1 + xe^x.$$

Let $f(x) = e^x$, $f'(x) = e^x$

f is continuous on $[0, x]$

differentiable on $(0, x)$

$$1 = \frac{x}{x-0} < \frac{e^x - 1}{x-0} < \frac{xe^x}{x-0} = e^x$$

By MVT, there exists $c \in (0, x)$ such that

$$\frac{e^x - 1}{x} = \frac{f(x) - f(0)}{x - 0} = f'(c) = e^c$$

$$0 < c < x$$

$$1 = e^0 < e^c < e^x$$

$$\therefore 1 < \frac{e^x - 1}{x} < e^x \qquad x < e^x - 1 < xe^x$$

$$1 + x < e^x < 1 + xe^x$$

11. By using L'Hôpital rule, find the following limits.

(a) $\lim\limits_{x\to 0} \dfrac{\ln(\cos x)}{x^2}$

(b) $\lim\limits_{x\to 0} \dfrac{e^x + e^{-x} - 2}{1 - \cos 2x}$

(c) $\lim\limits_{x\to \pi^+} \dfrac{\sin x}{\sqrt{x - \pi}}$

(d) $\lim\limits_{x\to 0^+} \dfrac{\ln(\cos 3x)}{\ln(\cos 2x)}$

(a) $\lim\limits_{x\to 0^+} x^2 \ln x$

(b) $\lim\limits_{x\to \frac{\pi}{2}} (2x - \pi) \sec x$

12. By using L'Hôpital rule, find the following limits.

(a) $\lim\limits_{x\to 0^+} \dfrac{\ln(\sin x)}{\ln(\tan x)}$

(b) $\lim\limits_{x\to 0^+} \dfrac{\ln(\tan 2x)}{\ln(\tan 3x)}$

$\cancel{\dfrac{x^2}{\ln x}}$

(c) $\lim\limits_{x\to \frac{\pi}{2}^-} \dfrac{4 \tan x}{1 + \sec x}$

(d) $\lim\limits_{x\to \infty} x^n e^{-ax}$, where n is a natural number and a is a positive real numb

13. By using L'Hôpital rule, find the following limits.

(a) $\lim\limits_{x\to 0^+} x^2 \ln x$ $= \lim\limits_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x^2}} = \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x}}{-\frac{2}{x^3}} = \lim\limits_{x\to 0^+} \dfrac{-x^2}{2} = 0$

$\left(\dfrac{-\infty}{\infty}\right)$

(b) $\lim\limits_{x\to \frac{\pi}{2}} (2x - \pi) \sec x$

(c) $\lim\limits_{x\to 1^+} (x^2 - 1) \tan \dfrac{\pi x}{2}$

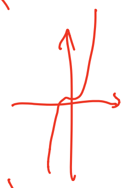
(d) $\lim\limits_{x\to \infty} x\left(\dfrac{\pi}{2} - \tan^{-1} x\right)$

$\lim\limits_{x\to 1^+} (x^2 - 1) \dfrac{\sin \frac{\pi x}{2}}{\cos \frac{\pi x}{2}}$

$$= \lim_{x\to 1^+} \sin\frac{\pi x}{2} \cdot \lim_{x\to 1^+} \frac{x^2-1}{\cos\frac{\pi x}{2}}$$

$$(\tan^{-1}x)' = \frac{1}{1+x^2}$$

$$\lim_{x\to\infty} x\left(\frac{\pi}{2} - \arctan x\right)$$



$(\arcsin x)'$

$(\arccos x)'$

$$= \lim_{x\to\infty} \frac{\frac{\pi}{2} - \arctan x}{\frac{1}{x}} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to\infty} \frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{x^2}{1+x^2} = \lim_{x\to\infty} \frac{1}{\frac{1}{x^2}+1} = \frac{1}{0+1} = 1$$

14. By using L'Hôpital rule, find the following limits.

(a) $\lim_{x\to 0}\left(\frac{1}{e^x-1} - \frac{1}{x}\right)$ $= \lim_{x\to 0} \frac{x-e^x+1}{x(e^x-1)}$

(b) $\lim_{x\to 0}\left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right)$ $= \lim_{x\to 0} \frac{x^2-\sin^2 x}{x^2\sin^2 x}$

15. By using L'Hôpital rule, find the following limits.

(a) $\lim_{x\to 0} x^x$

(b) $\lim_{x\to\infty} (e^{3x}-5x)^{1/x}$

(c) $\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$

(d) $\lim_{x\to 0} \sin x \ln(\sin x)$

Let $y = (e^{3x}-5x)^{\frac{1}{x}}$ (∞^0)

$$\ln y = \frac{\ln(e^{3x}-5x)}{x}$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(e^{3x}-5x)}{x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{1}{e^{3x}-5x}(3e^{3x}-5)}{1}$$

$$= \lim_{x\to\infty} \frac{3e^{3x}-5}{e^{3x}-5x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{9e^{3x}}{3e^{3x}-5}$$

$$= \lim_{x\to\infty} \frac{9}{3-\frac{5}{e^{3x}}} = 3$$

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y}$$

$$= e^{\lim_{x\to\infty} \ln y}$$

$$= e^3$$

$$\lim_{x\to\infty} (e^{3x}-5x)^{\frac{1}{x}}$$

$$= \lim_{x\to\infty} e^{\frac{1}{x}\ln(e^{3x}-5x)}$$

$$\lim_{x\to 0} \sin x \ln(\sin x)$$

$$= \lim_{x \to 0} \frac{\ln(\sin x)}{\csc x}$$

$$(\csc x)' = -\csc x \cot x$$

$$= \lim_{x \to 0} \frac{\frac{1}{\sin x} \cos x}{-\csc x \cot x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

$$= \lim_{x \to 0} -\sin x = 0$$